Self-excited oscillations of a third-order system[☆]

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Abstract

A third-order self-excited oscillatory system in the neighbourhood of a stable local integral manifold is investigated. A periodic manifold and a corresponding system in amplitude-phase variables are constructed with approximately the required accuracy. Using the procedure of separation of variables (averaging) the conditions for the existence, uniqueness and stability of self-excited oscillation modes are established, and critical cases of the degeneracy of these conditions are considered. A thermomechanical model of the self-excitation of oscillations inherent in gas-dynamic systems with a heat source is taken as an example. The bifurcation pattern of self-excited oscillations in the space of the governing parameters of the system is investigated in the second approximation. © 2006 Elsevier Ltd. All rights reserved.

Self-excited oscillations of systems with one degree of freedom have been investigated in detail in a number of theoretical (see Refs. 1–5, etc.) and applied (see Refs. 3,5–8, etc.) publications. The main mathematical techniques used to investigate them are different versions of the small-parameter method (perturbation theory) and qualitative (topological) phase-plane methods. The problem consists of determining the conditions for the existence, uniqueness and stability of limit cycles, and also an approximate calculation of the main characteristics of the self-excited oscillations (the amplitude, frequency, form of the limit cycles, etc.). A numerical-analytical method of accelerated convergence was recently used in Refs. 8 and 9 for the effective construction of the periodic motions of self-excited oscillatory systems for small and moderately large self-excitation coefficients.

Much fewer results have been obtained for self-excited oscillatory systems of higher order than the second. We note the investigations of quasi-linear self-excited oscillations of two coupled second-order radio circuits using the Lyapunov-Poincaré methods.[3] Quasi-linear self-excited oscillations of a Helmholtz resonator with a thermodynamic self-excitation mechanism were investigated in Ref. 10 for the three-dimensional Teodorchik model.[7] Using asymptotic methods of non-linear mechanics, a local integral manifold of the system was constructed in Refs. 11,12. It enables one, with the required degree of accuracy, to represent a self-excited oscillatory system in standard form for amplitude-phase variables and also enables the averaging method[13,14] to be used.

Third-order self-excited oscillatory systems are often encountered in nature and technology. We note the following phenomena: geysers, variable-brightness cepheid-type stars, self-excited oscillations in chemical and nuclear reactors, vapour "singing" in the pores of a thick layer of scum, and gas pulsations in the combustion chambers of jet engines. Investigations of such systems, both analytically and numerically, are of interest from the theoretical and applied points of view. The solution and analysis of such problems for small and moderately large self-excitation coefficients, leading to bifurcations of the self-excited modes of oscillation, is of considerable importance.

1. Formulation of the problem

Consider a three-dimensional non-linear self-excited oscillatory system containing a basic subsystem, namely, a second-order quasi-linear section and a one-dimensional self-excitation subsystem. We will assume that the equations of motion, in dimensionless variables, have the form

$$\ddot{x} + x = -\varepsilon\vartheta + \varepsilon f(x, \dot{x}), \quad \dot{\vartheta} + \kappa\vartheta = g(x, \dot{x}), \quad \kappa \geq \kappa_0 > 0 \tag{1.1}$$

Here $0 < \varepsilon \ll 1$ is a small parameter representing the effect of the excitation variable $\varepsilon\vartheta$ and the perturbations εf on the main variable x. For these ε the parameter κ and the function g (together with f) define a local integral manifold of system (1.1), to which it tends as $t \to \infty$, where t is the dimensionless time.

In the traditional case of a system with one degree of freedom we will assume $\vartheta = \kappa^{-1}g(x, \dot{x})$, and the quasi-linear self-excited oscillations are described by the first equation of (1.1). If $g = \kappa g^*$, $g^* \sim 1$, we have $\vartheta \to g^*$ as $\kappa \to \infty$ according to the theory of singularly perturbed differential equations [15].

For the general situation considered here ($\kappa \sim 1$) we will convert system (1.1) to standard form.[11–14] We will first represent it in the form of a third-order equation by differentiating the first equation of (1.1) with respect to t and eliminating ϑ, $\dot{\vartheta}$, so that we obtain the required relation

$$\dddot{x} + \kappa\ddot{x} + \dot{x} + \kappa x = \varepsilon F(x, \dot{x}, \ddot{x}), \quad F \equiv \kappa f + f'_x \dot{x} + f'_{\dot{x}} \ddot{x} - g \tag{1.2}$$

Note that this is quasi-linear, despite the considerable non-linearity of system (1.1). When $\varepsilon = 0$ the characteristic exponents $\lambda_{1,2,3}$ of Eq. (1.2) and its general solution x_0 have the form

$$\lambda_{1,2} = \pm i, \quad \lambda_3 = -\kappa; \quad x_0 = A_1 \cos t + A_2 \sin t + A_3 \exp(-\kappa t) \tag{1.3}$$

where $A_{1,2,3}$ are arbitrary constants. As $t \to \infty$ the solution x_0 (1.3) approaches the 2π-periodic function $x_* = a \sin\psi$, where a is the amplitude and $\psi = t + \psi_0$ is the phase; the constants a and ψ_0 are defined in a known way in terms of $A_{1,2}$.

To use the method of local integral manifolds[11,12] we will replace the variables $(x, \dot{x}, \ddot{x})$ by (a, ψ, z) in Eq. (1.2):

$$x = a \sin\psi + z, \quad \dot{x} = a \cos\psi - \kappa z, \quad \ddot{x} = -a \sin\psi + \kappa^2 z \tag{1.4}$$

The unknown z represents the local integral manifold of the system mentioned above.

Differentiating expressions (1.4), by virtue of the perturbed Eq. (1.2) we obtain the required system of standard form[13,14]

$$\dot{a} = \varepsilon A(a, \psi, z), \quad A = (\kappa\cos\psi - \sin\psi)F^*, \quad a \geq a_0 > 0$$

$$\dot{\psi} = 1 + \varepsilon\Psi(a, \psi, z), \quad \Psi = -a^{-1}(\kappa\sin\psi + \cos\psi)F^* \tag{1.5}$$

$$\dot{z} + \kappa z = \varepsilon Z(a, \psi, z), \quad Z = F^*, \quad F^* \equiv (1 + \kappa^2)^{-1}F$$

The function F^* (1.5) is obtained by substituting expressions (1.4) into the function F from (1.2). The right-hand sides of the equations are 2π-periodic in ψ and fairly continuous in a and z. If the condition $f = g = 0$ is satisfied when $x = \dot{x} = 0$, which is usually the case in applied problems, the function Ψ (and also the functions A and Z) will be regular for all $a \geq 0$. Otherwise, instead of the amplitude-phase variables (a, ψ) we will introduce osculating Van der Pol type variables (a, b) and make the replacement

$$x = a \sin t + b \cos t + z, \quad \dot{x} = a \cos t - b \sin t - \kappa z, \quad \ddot{x} = -a \sin t - b \cos t + \kappa^2 z \tag{1.6}$$

For (a, b) we obtain Bogolyubov standard equations. The advantages of these equations is the absence of singularities when $a, b \to 0$, if $f, g \neq 0$ when $x = \dot{x} = 0$, but a drawback is the increase in the order of the system: the right-hand sides depend on a, b, z and are 2π-periodic in t.

Further, for system (1.5) we will construct a local integral manifold, described by the expression $z(a, \psi, \varepsilon)$ with the required degree of accuracy in ε. It is substituted into the equations for the variables a, ψ, which define the possible

self-excited oscillations of the system. Further investigations are carried out by traditional asymptotic or regular small-parameter methods.[1,3,13,14] The averaging method enables us to simplify the construction of the approximate limit cycle considerably in $(x, \dot{x}, \ddot{x})$ space and to analyse the transient.

2. Construction of the local integral manifold

After a relatively short time $\Delta t = \kappa^{-1} \ln(|z^0|\varepsilon^{-1})$ system (1.5) arrives from an arbitrary state $z(0) = z^0$ to an ε-neighbourhood of the state $z = 0$ and performs motions in the neighbourhood of the manifold $z = \varepsilon h(a, \psi, \varepsilon)$.[11–13] The unknown h is a 2π-periodic function in ψ and regular in a, ε, and is constructed as the solution of a first-order partial differential equation, which we will denote by primes with corresponding subscripts.

We have

$$h'_\psi + \kappa h = H(a, \psi, [z]), \quad H = Z - \varepsilon(h'_\psi \Psi + h'_a A), \quad z = \varepsilon h \tag{2.1}$$

where A, Ψ and Z are functions of the unknown h and specified ψ, a and ε in the permissible region of their variation. For sufficient smoothness with respect to a of these functions the unknown h can be obtained with a required degree of accuracy in ε by the asymptotic expansion

$$h = h_1(a, \psi) + \varepsilon h_2(a, \psi) + \varepsilon^2 h_3(a, \psi) + \dots \tag{2.2}$$

The coefficients h_j $(j = 1, 2, \dots)$ are defined as 2π-periodic solutions of a sequence of inhomogeneous linear equations, the right-hand sides of which are calculated recurrently using the known expressions for $h_1, h_2, \dots, h_{j-1}$ and the corresponding coefficients of the expansions of the functions A, Ψ and Z after substituting expression (2.2) into Eq. (2.1). As a result we have equations which determine the required h_j

$$h'_{j,\psi} + \kappa h_j = H^*_j(a, \psi), \quad H^*_1 = Z(a, \psi, 0)$$

$$H^*_2 = Z'_{z0} - [Z_0 - \kappa h^*_1]\Psi_0 - h^{*'}_{1,a}A_0, \dots \tag{2.3}$$

Here and henceforth the zero subscript denotes that the value of the function is taken for $z = 0$. The coefficients H^*_3, H^*_4 are calculated similarly using the quantities $h^*_1, h^*_2, \dots, h^*_{j-1}$ obtained. The required 2π-periodic solution $h^*_j(a, \psi)$ of Eq. (2.3) can be represented in the form of an improper integral, or a trigonometric series[11–13]

$$h^*_j(a, \psi) = \int_{-\infty}^{\psi} \exp[-\kappa(\psi - \varphi)]H^*_j(a, \varphi)d\varphi =$$

$$= \sum_{n=0}^{\infty} (n^2 + \kappa^2)^{-1}[(\kappa H^{*c}_{jn} - nH^{*s}_{jn})\cos n\psi + (nH^{*c}_{jn} + \kappa H^{*s}_{jn})\sin n\psi] \tag{2.4}$$

If the function F turns out to be a polynomial in $x, \dot{x}$, the expansion of the function H^*_j in a trigonometric series contains a finite number $N = N(j)$ of terms $H^{*c,s}_{j,n}$, which increases as the subscript j increases.

Together with the standard procedure for the expansion of the unknown variable h in powers of ε (2.2), to find the periodic solution of Eq. (2.1) one can use the recurrence method of successive approximations equivalent to it

$$h^*_{(j)}(a, \psi, \varepsilon) = \int_{-\infty}^{\psi} \exp[-\kappa(\psi - \varphi)]H^*_{(j)}(a, \varphi, \varepsilon)d\varphi =$$

$$= \sum_{n=0}^{\infty} (n^2 + \kappa^2)^{-1}[(\kappa H^{*c}_{(j)n} - nH^{*s}_{(j)n})\cos n\psi + (nH^{*c}_{(j)n} + \kappa H^{*s}_{(j)n})\sin n\psi] \tag{2.5}$$

$$H^*_{(1)} = H(a, \psi, 0), \dots, H^*_{(j)} = H(a, \psi, \varepsilon[h^*_{(j-1)}(a, \psi)]), \quad j = 1, 2, \dots$$

Here $H^{*c,s}_{(j)n}$ are the Fourier coefficients of the function $H^*_{(j)}$, which is constructed using previous approximations $h^*_{(1)}, h^*_{(2)}, \dots, h^*_{(j-1)}$ and is 2π-periodic in ψ. Note that when constructing the solution in the form (2.2)–(2.5), the

order of the derivatives with respect to the variable a of the functions A, Ψ, Z, i.e. of the initial function F (1.2) with respect to the variables x, $\dot{x}$, increases. Expansion (2.2) or the recurrence procedure (2.5), generally speaking, diverges as $j \to \infty$, and the solution has an asymptotic form. Its accuracy in powers of the small parameter ε is determined by the degree of smoothness of the functions in the variable a.

After substituting the expression obtained $z = \varepsilon h^*$ into the equation for the amplitude a and the phase ψ (1.5), we have a standard system of the form

$$
\begin{aligned}
\dot{a} &= \varepsilon A^*(a, \psi, \varepsilon), \quad A^*(a, \psi, \varepsilon) \equiv A(a, \psi, \varepsilon h^*) \\
\dot{\psi} &= 1 + \varepsilon \Psi^*(a, \psi, \varepsilon), \quad \Psi^*(a, \psi, \varepsilon) \equiv \Psi(a, \psi, \varepsilon h^*)
\end{aligned}
\tag{2.6}
$$

The solutions a, ψ, z of Eqs. (1.5) tend exponentially with time to the solutions a^*, ψ^*, εh^* of system (2.6) for the same initial data.[11] Hence, the problem of the self-excited oscillations of the initial system (1.1) (or the modified systems (1.2) and (1.5)) reduces to an investigation of the comparatively simple autonomous system (2.6), which allows of the use of standard analytical and asymptotic small-parameter methods. Note that the analytical Lyapunov-Poincaré methods[3] enable one to establish the existence and stability conditions and give a constructive procedure for constructing the stationary periodic solution of Eq. (2.6). Moreover, they are applicable both to the initial Eq. (1.1) and to the modified Eqs. (1.2), (1.5). The Krylov-Bogolyubov averaging method[13,14] enables system (2.6) to be simplified, enables the variables a and ψ to be separated and enables one to determine the stationary points for the amplitude equation, corresponding to self-excited oscillations, and also enables the transients to be investigated.

3. Approximate solution of the problem of self-excited oscillations

We will apply an asymptotic procedure for separating the variables a and ψ to system of Eq. (2.6).[13,14] More accurately, we will construct the approximate replacement (a, ψ) by (α, φ) such that the right-hand sides of the equations for the "averaged" amplitude α and phase φ will be independent of the fast variable φ. These replacements and equations are constructed in the form of asymptotic expansions in powers of ε^j ($j = 0, 1, 2, \ldots$)

$$
\begin{aligned}
a &= \alpha + \varepsilon u_1(\alpha, \varphi) + \varepsilon^2 u_2 + \ldots + \varepsilon^j u_j + \ldots, \quad u_j(\alpha, \varphi + 2\pi) \equiv u_j(\alpha, \varphi) \\
\psi &= \varphi + \varepsilon v_1(\alpha, \varphi) + \varepsilon^2 v_2 + \ldots + \varepsilon^j v_j + \ldots, \quad v_j(\alpha, \varphi + 2\pi) \equiv v_j(\alpha, \varphi)
\end{aligned}
\tag{3.1}
$$

$$
\begin{aligned}
\dot{\alpha} &= \varepsilon \Lambda_1(\alpha) + \varepsilon^2 \Lambda_2 + \ldots + \varepsilon^{j+1} \Lambda_{j+1} + \ldots \\
\dot{\varphi} &= 1 + \varepsilon \Phi_1(\alpha) + \varepsilon^2 \Phi_2 + \ldots + \varepsilon^{j+1} \Phi_{j+1} + \ldots
\end{aligned}
\tag{3.2}
$$

The unknown functions u_j, v_j together with Λ_{j+1}, Φ_{j+1} are constructed recurrently after substituting replacement (3.1) into Eq. (2.6) and taking into account representations (3.2) for $\dot{\alpha}$, $\dot{\varphi}$. By equating coefficients of like powers of ε^j we can successively obtain Λ_1, Φ_2, $\ldots$, Λ_{j+1}, Φ_{j+1} as the mean over φ of the corresponding known expressions, and also determine the unknowns u_1, v_1, $\ldots$, u_j, v_j. In particular

$$
\Lambda_1(\alpha) = \langle A_1(\alpha, \varphi) \rangle, \quad \Phi_1(\alpha) = \langle \Psi_1(\alpha, \varphi) \rangle
$$

$$
u_1(\alpha, \varphi) = \int_0^\varphi (A_1(\alpha, \varphi_1) - \langle A_1 \rangle) d\varphi_1, \quad v_1(\alpha, \varphi) = \int_0^\varphi (\Psi_1(\alpha, \varphi_1) - \langle \Psi_1 \rangle) d\varphi_1
\tag{3.3}
$$

$$
\begin{aligned}
\Lambda_2(\alpha) &= \langle A'_{1\alpha} u_1 \rangle + \langle A'_{1\varphi} v_1 \rangle + \langle A_2 \rangle - \langle u'_{1\alpha} \rangle \Lambda_1 \\
\Phi_2(\alpha) &= \langle \Psi'_{1\alpha} u_1 \rangle + \langle \Psi'_{1\varphi} v_1 \rangle + \langle \Psi_2 \rangle - \langle v'_{1\alpha} \rangle \Lambda_1
\end{aligned}
\tag{3.4}
$$

The angular brackets denote averaging over the phase φ. The functions A_j and Ψ_j are obtained by substituting $z = \varepsilon h^*$ into the expressions for A^*, Ψ^* and expanding in powers of ε^j

$$A^*(a, \psi, \varepsilon h^*(a, \psi, \varepsilon)) = A_1(a, \psi) + \varepsilon A_2 + \ldots + \varepsilon^j A_{j+1} + \ldots, \quad A_1 \equiv A^*(a, \psi, 0)$$

$$\Psi^*(a, \psi, \varepsilon h^*(a, \psi, \varepsilon)) = \Psi_1(a, \psi) + \varepsilon \Psi_2 + \ldots + \varepsilon^j \Psi_{j+1} + \ldots, \quad \Psi_1 \equiv \Psi^*(a, \psi, 0)$$

(3.5)

Further expansions of the functions A_j and Ψ_j (3.5) involve substituting a and ψ (3.1), which occurs in expressions (3.4).

The functions $\Lambda_1(\alpha)$, $\Phi_1(\alpha)$ determine the system of Eq. (3.2) of the first approximation (terms $O(\varepsilon^2)$ are ignored). They enable us to calculate the required unknowns a and ψ with an error $O(\varepsilon)$, and also $z = \varepsilon h$ with an error $O(\varepsilon^2)$ in the section $t_1 \leq t \leq L\varepsilon^{-1}$, where $t_1 \sim \ln(\varepsilon^{-1})$, for system (1.5). As a result we will obtain the variables x, $\dot{x}$, $\ddot{x}$ or ϑ. For the solution of the first approximation it is unnecessary to construct the functions u_1 and v_1.

The solution of the second approximation with error $O(\varepsilon^2)$ in the section $2t_1 \leq t \leq L\varepsilon^{-1}$ requires integration of Eq. (3.2) taking the terms $\varepsilon^2 \Lambda_2$, $\varepsilon^2 \Phi_2$ into account (terms $O(\varepsilon^3)$ are dropped). This is also done by standard methods: separation of the variables, expansion in powers of ε, numerical calculations in a short interval of the slow time $\tau = \varepsilon t$, $\tau_1 \leq \tau \leq L$ etc. For these constructions one can substitute expressions of the first approximation into terms of the order of ε^2 and into the functions u_1 and v_1.

To obtain and investigate the self-excited oscillations, it is of particular interest to determine the stationary points $\alpha^*_{(j)}$ of the first Eq. (3.2) for α in the j-th approximation in ε

$$P_{(j)}(\alpha, \varepsilon) = \Lambda_1(\alpha) + \varepsilon \Lambda_2 + \ldots + \varepsilon^j \Lambda_{j+1} = 0, \quad \alpha > 0$$

$$\alpha^*_{(j)}(\varepsilon) = \arg_\alpha P_{(j)}(\alpha, \varepsilon), \quad \alpha^*_{(j)} > 0$$

(3.6)

The simple sufficient conditions for the existence, uniqueness and stability of self-excited oscillations for fairly small $\varepsilon > 0$ are obtained by analysing the roots of (3.6) of the first approximation, i.e. for $\varepsilon = 0$

$$\alpha^*_{(1)} = \alpha^*_{(1)}(0) = \arg_\alpha \Lambda_1(\alpha), \quad \alpha > 0$$

(3.7)

We will assume that the required root $\alpha^*_{(1)}$ (3.7) exists and is simple: $\Lambda'_1(\alpha^*) \neq 0$. Then, for sufficiently small $\varepsilon > 0$ Eq. (3.6) allows of a ε-close $\alpha^*_{(1)}$ root $\alpha^*_{(j)}(\varepsilon)$. It can be calculated with the required degree of accuracy with respect to ε by expansions or successive approximations. The self-excited oscillations of the initial system for which the limit cycles are constructed in the phase space x, $\dot{x}$, $\ddot{x}$, for example, in the form of projections onto the $(x, \dot{x})$ plane, etc., correspond to it. The periodic solution, i.e. the self-excited oscillations, will be asymptotically orbitally stable or Lyapunov stable if $\Lambda'_1(\alpha^*) < 0$ and unstable in the case of the opposite inequality. Eq. (3.6) may allow of several simple roots; the above conclusions hold for each of them.

The situation when the root α^* is k-multiple, i.e. the following relations hold for it

$$\Lambda_1 = \Lambda'_1 = \ldots = \Lambda_1^{(k-1)} = 0, \quad \Lambda_1^{(k)}(\alpha^*) \neq 0, \quad k = 1, 2, \ldots$$

(3.8)

requires an additional investigation based on methods of the theory of implicit functions (Weierstrass' theorem[16]). In particular, if $k = 2$, the required quadratic root $\alpha^*_{(j)}(\varepsilon)$ is constructed as a function of ε in the form of expansions in powers of $\delta = \varepsilon^{1/2}$

$$\alpha^*_{(j)}(\varepsilon) = \alpha^* + \delta \alpha_1 + \delta^2 \alpha_2 + \ldots$$

(3.9)

The standard procedure reduces to algebraic equations, relating the unknown coefficients α_m ($m = 1, 2, \ldots$). They can be solved successively with corresponding fairly obvious assumptions (see below)

$$\alpha^*_1 = \pm(-2\Lambda_2/\Lambda''_1)^{1/2} \neq 0, \quad \alpha^*_2 = (\Lambda_2 \Lambda'''_1/3\Lambda''_1 - \Lambda'_1)/\Lambda''_1, \ldots$$

(3.10)

Expressions (3.10) indicate the splitting of the generating solution into a quantity $O(\sqrt{\varepsilon})$. If the known conditions break down, additional investigations are required. In the general case of a k-multiple root (3.8), the expansions are

carried out in powers of the small parameters δ_i (Refs. 3,16)

$$\alpha^*_{(j)}(\varepsilon) = \alpha^* + \delta_l \alpha_{l1} + \delta_l^2 \alpha_{l2} + \ldots, \quad \delta_l = \varepsilon^{1/k_l}, \quad \sum_{l=1}^{l^*} k_l \leq k \tag{3.11}$$

The integer (different) denominators k_l in relations (3.11), where $1 \leq k_l \leq k$, $l = 1, 2, \ldots, l^*$, which define fractional powers of the small parameter, can be found in the standard way.[16] For small values of $k = 2, 3, \ldots$, in applied calculations it is assumed that $k_l = 1/2, 1/3, \ldots$ and the problem of the compatibility (solvability) of the equations for unknown α_{l1}, $\alpha_{l2}, \ldots$ is investigated. In this case, analogous to (3.10), non-uniqueness of the solution ("splitting") may occur.

The situation when $\Lambda_1(\alpha) \equiv 0$ is of particular interest in theoretical and applied aspects. In this case, the governing equation is $\Lambda_2(\alpha) = 0$. An analysis of the conditions for the existence, uniqueness and stability of self-excited oscillations is carried out in the same way as above.

Note that an investigation of the problems of the existence, uniqueness, stability and qualitative behaviour of a self-excited oscillatory system is usually carried out using the system of the first approximation (3.2), in which terms $O(\varepsilon^2)$ are dropped. By introducing the slow argument $\tau = \varepsilon t$, the main equation (for α) is reduced to a form that is independent of the important parameter ε, which defines the self-excitation quantity. For applied problems, however, the behaviour of the system for moderately large specific values of ε, rather than for infinitesimal values, may be of interest. Then, the solution constructed can be used as the initial approximation in numerical-analytical procedures of continuation with respect to the parameter.[9] An increase in ε in problems of the class considered often leads to bifurcations of the self-excited oscillations.

For a preliminary local investigation of the possible evolution of the oscillation modes, it may turn out to be useful to construct and analyse a solution taking into account higher powers of ε, i.e. terms $\varepsilon^2 \Lambda_2$, $\varepsilon^3 \Lambda_3$, etc. in expansions (3.2) and a change in ε over a wider range (of the order of unity). Note that an increase in the values of the parameter ε and "escape" beyond the limits of the region, formally assigned by the methods of asymptotic theory or regular expansions, has an extremely conditional form of directing considerations. This approach is used below to investigate thermomechanical self-excited oscillations of a system based on solution of the second approximation of the asymptotic methods of local integral manifolds and averaging (see above).

4. Self-excited oscillations in thermomechanical systems (the second approximation)

As an example of the use of the procedure for the asymptotic analysis of quasi-linear third-order systems described above, we will consider the problem of the occurrence of self-excited oscillations in thermomechanical systems. The governing Eq. (1.2) has the following form in dimensionless variables

$$\dddot{\xi} + (\kappa + \varepsilon\sigma)\ddot{\xi} + (1 + \varepsilon\sigma\kappa)\dot{\xi} + (\kappa + \varepsilon)\xi - \varepsilon\xi^3 = 0, \quad \varepsilon \ll 1 \tag{4.1}$$

In particular, the system describing the self-excited oscillations of a Helmholtz resonator[7,10] reduces to an equation of the form (4.1). The expressions $f = -\sigma\dot{\xi}$, $g = \xi - \xi^3$ of system (1.1) correspond to Eq. (4.1).

We will change to a system of equations in standard form (1.5) using formulae (1.4), in which now $x = \xi$. As a result we obtain the following expressions for the right-hand sides of system (1.5)

$$A = (\kappa\cos\psi - \sin\psi)Z(\xi, \dot{\xi}, \ddot{\xi})$$

$$\Psi = -a^{-1}(\kappa\sin\psi + \cos\psi)Z(\xi, \dot{\xi}, \ddot{\xi}), \quad a > 0 \tag{4.2}$$

$$Z \equiv (1 + \kappa^2)^{-1}(\xi^3 - \xi - \sigma\kappa\dot{\xi} - \sigma\ddot{\xi})$$

Changing from the function z to the function h, using relations (2.1) and (2.2), we can simultaneously obtain expressions for the first two terms of the expansion of h in series of perturbation theory. Then, from the calculated

expressions for H_1^* and H_2^*, the first two approximations for h can be found using formulae (2.5)

$$h_1 = \frac{a}{4(\kappa^2 + 1)}\left\{\frac{1}{\kappa^2 + 1}[\kappa(3a^2 - 4)\sin\psi - (4\sigma(\kappa^2 + 1) + 3a^2 - 4)\cos\psi] + \right.$$

$$\left. + \frac{a^2}{\kappa^2 + 9}[3\cos 3\psi - \kappa\sin 3\psi]\right\} \tag{4.3}$$

$$h_2 = \frac{a}{(\kappa^2 + 1)^4(\kappa^2 + 9)}\left\{h_{2,1}^s\sin\psi + h_{2,1}^c\cos\psi + \right.$$

$$\left. + \frac{\kappa^2 + 1}{\kappa^2 + 9}[h_{2,3}^s\sin 3\psi + h_{2,3}^c\cos 3\psi] + \frac{(\kappa^2 + 1)^2}{\kappa^2 + 25}[h_{2,5}^s\sin 5\psi + h_{2,5}^c\cos 5\psi]\right\} \tag{4.4}$$

The coefficients of the trigonometric polynomials for h_2, according to relations (2.2) and (2.4), have the form

$$h_{2,1}^s = \frac{1}{8}\{9(5\kappa^4 + 30\kappa^2 - 7)a^4 + (6(\kappa^2 + 1)(\kappa^4 + 18\kappa^2 - 15)\sigma - 72(\kappa^4 + 7\kappa^2 + 2))a^2 - $$

$$- 8(\kappa^2 + 9)((\kappa^4 - 1)\sigma - (3\kappa^2 - 1))\}$$

$$h_{2,1}^c = \kappa\{9(\kappa^2 + 5)a^4 + (3(\kappa^2 + 1)(\kappa^2 + 17)\sigma - 3(5\kappa^2 + 29))a^2 + (\kappa^2 + 1)(\kappa^2 + 9)\sigma^2 - $$

$$- 4(\kappa^2 + 1)(\kappa^2 + 9)\sigma + 4(\kappa^2 + 9)\}$$

$$h_{2,3}^s = -\frac{1}{16}a^2\{3(15\kappa^4 - 10\kappa^2 - 153)a^2 + 4[(\kappa^2 + 1)(\kappa^4 - 30\kappa^2 - 135)\sigma - $$

$$- 4(3\kappa^4 - 5\kappa^2 - 36)]\}$$

$$h_{2,3}^c = \frac{3}{2}\kappa a^2\{3(3\kappa^2 + 11)a^2 + 2[(\kappa^4 + 4\kappa^2 + 3)\sigma - (5\kappa^2 + 17)]\}$$

$$h_{2,5}^s = \frac{9}{16}a^4(\kappa^2 - 15), \quad h_{2,5}^c = -\frac{9}{2}\kappa a^4$$

We will investigate the first and second approximations of the functions of the amplitude and phase, to analyse the behaviour of which it is sufficient to know the expression for h to terms of the first order of smallness, i.e. taking the expression for h_1 (4.3) into account.

Using system (2.6), after determining h_1 from system (1.5), we analyse a standard second-order system with rotating phase. Substituting the expansions of the amplitude and phase, and also their derivatives (3.1) and (3.2) into system (2.6) and separating the orders of perturbation theory and averaging, we successively obtain expressions for $\Lambda_{1,2}$ and $\Phi_{1,2}$

$$\Lambda = \varepsilon\Lambda^{(2)} + O(\varepsilon^3), \quad \Lambda^{(2)} = \Lambda_1 + \varepsilon\Lambda_2 = -\frac{a_\lambda\alpha^4 + 2b_\lambda\alpha^2 + c_\lambda}{2(\kappa^2 + 1)^3}\alpha \tag{4.5}$$

$$\Phi = \varepsilon\Phi^{(2)} + O(\varepsilon^3), \quad \Phi^{(2)} = \Phi_1 + \varepsilon\Phi_2 = -\frac{a_\varphi\alpha^4 + 2b_\varphi\alpha^2 + c_\varphi}{256(\kappa^2 + 1)^3(\kappa^2 + 9)} \tag{4.6}$$

The coefficients $a_{\lambda,\varphi}$, $b_{\lambda,\varphi}$, $c_{\lambda,\varphi}$ in the expressions for $\Lambda^{(2)}$ (4.5) and $\Phi^{(2)}$ (4.6) depend on ε, κ and σ, and are given by

$$a_\lambda = \frac{3}{16}\frac{\varepsilon\kappa[(\kappa^2+13)^2-48]}{\kappa^2+9}, \quad a_\varphi = 3\varepsilon(17\kappa^6+259\kappa^4+851\kappa^2-159)$$

$$b_\lambda = \frac{3}{16}[2(\kappa^2+1)^2-\varepsilon\kappa(\kappa^2+13)+4\varepsilon\kappa\sigma(\kappa^2+1)], \quad b_\varphi = 24(\kappa^2+9)(b_0\varepsilon+b_1)$$

$$b_0 = (3\kappa^2-1)(\kappa^2+1)\sigma-(2\kappa^4+11\kappa^2-3), \quad b_1 = 2\kappa(\kappa^2+1)^2$$

$$c_\lambda = -[(\kappa^2+1)^2-2\varepsilon\kappa-\sigma(\kappa^2+1)^3+\varepsilon\kappa\sigma(\kappa^2+1)], \quad c_\varphi = 32(\kappa^2+9)(c_0\varepsilon-c_1)$$

$$c_0 = (\kappa^2+1)^3\sigma^2-2(\kappa^2-1)(\kappa^2+1)\sigma+(\kappa^4+6\kappa^2-3), \quad c_1 = 4\kappa(\kappa^2+1)^2$$

A comparison of the expressions for the functions Φ and Λ, calculated to the first and second orders of smallness, shows that the inclusion of second-order terms leads to a change in the degree of the polynomial dependence of the right-hand sides of the derivative of the amplitude from the third to the fifth, and of the phase from the second to the fourth. It turns out then that the coefficients of higher powers of α are proportional to the parameter ε. Hence, the use of analytical procedures of expansion in ε to investigate the roots α of the equation

$$\Lambda^{(2)}(\alpha, \kappa, \sigma, \varepsilon) = 0, \quad \alpha = \alpha^*(\kappa, \sigma, \varepsilon) \tag{4.7}$$

i.e. to analyse the stationary points of system (2.6), leads to the possibility of constructing roots of Eq. (4.7) that are regular in ε. The use of the methods of perturbation theory in this case is complicated by the dependence of the coefficient of polynomial (4.5) on the other parameters of the problem, when the smallness of ε is compensated by the other parameters. Hence, the roots of Eq. (4.7) were analysed accurately.

After constructing and investigating the roots of the functions $\alpha^*(\kappa, \sigma, \varepsilon)$ (4.7), the stationary addition to the average frequency can be determined, i.e. the function

$$\Phi^{(2)}(\alpha^*(\kappa, \sigma, \varepsilon), \kappa, \sigma, \varepsilon) = \text{const} \tag{4.8}$$

In the space of the parameters κ, σ, ε one can investigate the behaviour of the function $\Phi^{(2)}$. For applications the sign-definiteness of $\Phi^{(2)}$ (4.8) is of considerable interest, and also the finding of the values of the parameters for which $\Phi^{(2)} = 0$ or $\Phi^{(2)} \lessgtr 0$.

We will analyse the behaviour of system (2.6) in the simpler case when there is no dissipation $\sigma (\sigma = 0)$, and we will determine the regions in which self-excited oscillations exist. To do this we will construct a pattern of the change in the types of dependence of $\Lambda^{(2)}$ and $\Phi^{(2)}$ on α when the parameters ε and κ vary.

When $\sigma = 0$ the parameters ε and κ are grouped in the function $\Lambda^{(2)}$ in such a way that the pattern becomes "quasi-uniform" (i.e. it depends implicitly on the parameter κ) and can be reduced to the form shown in Fig. 1. In this case,
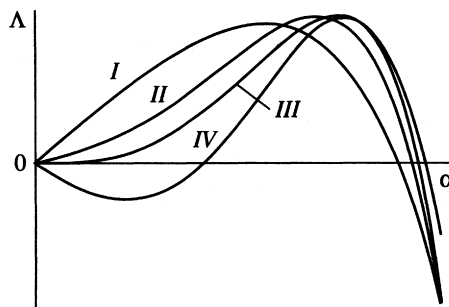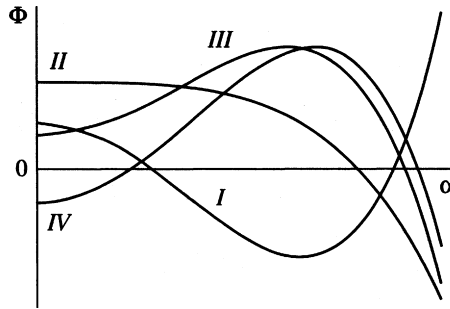

Fig. 1.

Fig. 2.

the limits of the oscillation modes are determined by the parameters $\varepsilon_{\lambda, b}$ and $\varepsilon_{\lambda, c}$, the values of which depend on the parameter κ

$$\varepsilon_{\lambda, b} = \frac{2(\kappa^2 + 1)^2}{\kappa(\kappa^2 + 13)}, \quad \varepsilon_{\lambda, c} = \frac{(\kappa^2 + 1)^2}{2\kappa}, \quad 0.54387 \le \varepsilon_{\lambda, c}^{(\min)} < 0.54389$$

An analysis of the stationary points of the oscillation amplitude, carried out using (4.7), shows that in the range of variation of ε from 0 to $\varepsilon_{\lambda, b}$ the unique stable point of self-excited oscillations is the point with $\alpha = \alpha_{\lambda, 1}$, when $\Lambda^{(2)}$ vanishes (curve I).

On transferring to the next interval up to the point $\varepsilon_{\lambda, c}$, the derivative of $\Lambda^{(2)}$ at zero decreases, and the equilibrium position becomes weakly unstable. In the neighbourhood of this point it is possible for oscillatory modes to exist over a limited time interval, after which an increase in the perturbations will lead to a loss of stability (curves II and III). A further increase in ε ($\varepsilon > \varepsilon_{\lambda, c}$) leads to the occurrence of two stable equilibrium positions $\alpha = 0$ and $\alpha = \alpha_{\lambda, 2}$, in the neighbourhood of which self-excited oscillations occur, and one more unstable stationary point $\alpha = \alpha_{\lambda, 1}$ appears (curve IV).

In order to investigate the nature of the change in the average frequency of the oscillations, we will analyse the pattern of the qualitative behaviour of the function $\Phi^{(2)}$, by first noting that, unlike the graphs for the amplitude, the qualitative dependence of the oscillation frequency is actually two-dimensional. In Fig. 2 we show the permissible forms of the fourth-order curves $\Phi^{(2)}(\alpha)$. The regions in Fig. 3, bounded by the curves $\varepsilon_{\varphi, b}$ (the curve with number 1) and $\varepsilon_{\varphi, c}$ (the curve with number 2) correspond to the numbers of the curves in Fig. 2. It can be seen that the boundaries of the regions in which the modes of the oscillation frequency change are determined by three values of the parameter κ and the two curves of $\varepsilon_{\varphi, b}$ and $\varepsilon_{\lambda, c}$, determined from the results of an analysis of the behaviour of the biquadratic function $\Phi^{(2)}(\alpha)$, in the form

$$\varepsilon_{\varphi, b} = \frac{2\kappa(\kappa^2 + 1)^2}{2\kappa^4 + 11\kappa^2 - 3}, \quad \varepsilon_{\varphi, c} = \frac{4\kappa(\kappa^2 + 1)^2}{\kappa^4 + 6\kappa^2 - 3}$$

$$\kappa_0 = \arg(a_\varphi(\kappa)) \approx 0.42092, \quad \kappa_1 = \sqrt{\sqrt{145} - 11}/2 \approx 0.51029, \quad \kappa_2 = \sqrt{2\sqrt{3} - 3} \approx 0.68125$$
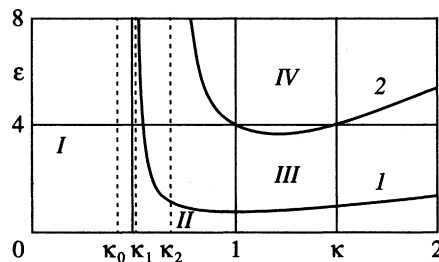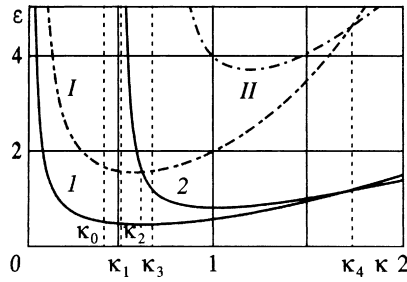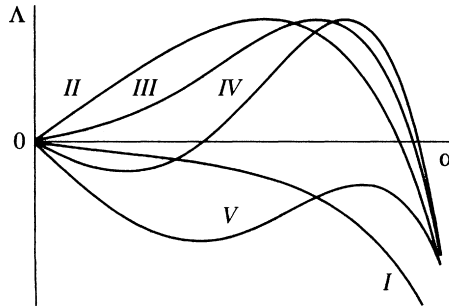

Fig. 3.

Fig. 4.

Fig. 5.

The straight lines $\kappa = \kappa_1$ and $\kappa = \kappa_2$ are the vertical asymptotes of the hyperbolae $\varepsilon_{\varphi,b}$ and $\varepsilon_{\varphi,c}$ respectively.

Comparing the position of the stationary points with the position of the zeros of the function $\Phi^{(2)}$, we can determine the corrections to the frequency of the oscillations compared with the unperturbed state. To do this we additionally set boundaries of the regions with qualitatively different behaviour of the functions $\varepsilon_{\varphi,b}$, $\varepsilon_{\varphi,c}$, $\varepsilon_{\lambda,b}$, $\varepsilon_{\lambda,c}$ (curves 1, I, 2, II respectively) on a single diagram (Fig. 4). The points of intersection of the boundary curves are denoted by $\kappa_3(\kappa_3 = \sqrt{-7\sqrt{73}}/2 \approx 0.62128)$ and $\kappa_4(\kappa_4 = \sqrt{3})$.

Analysis shows that the oscillation frequency in the neighbourhood of the second stable stationary point will always be reduced: $\Phi_{\alpha_{\lambda_1}}^{(+)} < 0$ (the superscript plus denotes that this quantity relates to the stable stationary point, and the plus-minus refers to a slightly unstable point); in the neighbourhood of the quasi-stable zero it will always increase: $\Phi_0^{(\pm)} > 0(\varepsilon_{\lambda,b}^{(0)} < \varepsilon < \varepsilon_{\lambda,c}^{(0)}, \kappa < \kappa_4)$; finally, for large values of $\varepsilon > \varepsilon_{\lambda,c}^{(0)}$ and $\kappa < \kappa_4$ the zero becomes a stable point, and the oscillation frequency in its neighbourhood will be reduced: $\Phi_0^{(+)} < 0$.

When the dissipation σ is taken into account, expressions (4.5) and (4.6) lead to a considerable complication of the patterns of the rates of change of the amplitudes and phase of the oscillations. Oscillation modes arise which were not
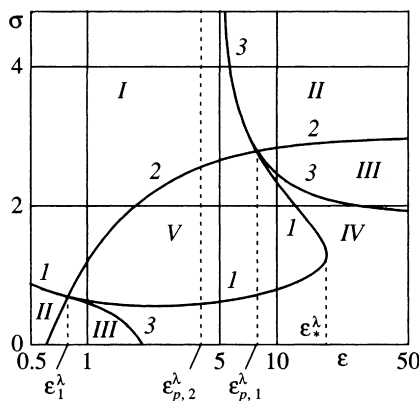

Fig. 6.

present in the previous case. The pattern of the change in the oscillation rates becomes "quasi-two-dimensional" (the limits of the oscillation modes turn out to be dependent on the parameter κ), while the frequencies are three-dimensional. The results of an analysis of the patterns of the rates of change of the amplitudes are shown in Figs. 5 and 6.

The boundaries of the oscillation modes for the patterns of the rates of change of the oscillation amplitudes in the second approximation $\Lambda^{(2)}(\alpha)$ for $\sigma \neq 0$ are defined by the quantities

$$\sigma_j^\lambda = \frac{(y^2 + 21y + 72)y^2 + (y^2/2 + 6(y+8) + (-1)^j\sqrt{(y^2 + 24y + 96)\Delta})\varepsilon\kappa}{6y(y+8)\varepsilon\kappa}, \quad j = 1, 2$$

$$\sigma_p^\lambda = \frac{y^2 - 2\varepsilon\kappa}{y(y^2 - \varepsilon\kappa)}, \quad \sigma_y^\lambda = \frac{(y+12)\varepsilon\kappa - 2y^2}{4\varepsilon\kappa y}, \quad \varepsilon_{p,1}^\lambda = 2\varepsilon_{p,2}^\lambda = \frac{(\kappa^2 + 1)^2}{\kappa}$$

$$\varepsilon_1^\lambda = \frac{4y}{(y + 10 - \sqrt{y^2 + 12y + 68})\kappa}$$

$$\varepsilon_*^\lambda = y^2 \frac{\sqrt{3}\sqrt{(y+8)(3y^3 + 44y^2 + 108y + 96)} - y^2 - 6y - 48}{2(y^2 + 18y + 48)\kappa}$$

$$y = \kappa^2 + 1, \quad \Delta = \frac{1}{(\varepsilon\kappa)^2}[(y^2 + 18y + 48)y^4 + (y^2 + 6y + 48)y^2\varepsilon\kappa - 2(y+6)y(\varepsilon\kappa)^2]$$

The permissible forms of the relations $\Lambda^{(2)}(\alpha)$ are denoted by Roman numerals in Fig. 5, and in Fig. 6 they indicate the regions of values of the parameters (σ, ε) corresponding to them. The boundaries of the oscillation modes are denoted in Fig. 6 by Arabic numerals: the two single-valued curves $\sigma_{1,2}^\lambda$, curves $2 - \sigma_y^\lambda$ and curve $3 - \sigma_p^\lambda$ correspond to the two-valued curve 1.

Further investigations confirm that, in the third-order self-excited oscillatory system (4.1) considered, bifurcation of the steady oscillation modes is possible when the self-excitation parameter ε increases. Preliminary calculations showed that when $\varepsilon \approx 1$, $\kappa \approx 0.5$, $\sigma = 0$, bifurcation occur in the neighbourhood of the value $\alpha \approx \alpha^* = \sqrt{4/3}$, corresponding to quasi-linear self-excited oscillations.[10] Analysis of the oscillation modes over a wide range of variation of the parameter ε for different values of κ and σ requires the development of a highly accurate algorithm for constructing periodic solutions in combination with the procedure of continuation with respect to the parameters.[8–10]

Acknowledgement

References

1. Andronov AA, Vitt AA, Khaikin SE. *Theory of Oscillations*. Moscow: Fizmatgiz; 1959.
2. Andronov AA, Leontovich YeA, Gordon II, Maier AG. *A Qualitative Theory of Second-Order Dynamical Systems*. Moscow: Nauka; 1966.
3. Malkin IG. *Some Problems of the Theory of Non-linear Oscillations*. Moscow: Gostekhizdat; 1956.
4. Lefschetz S. *Differential Equations: Geometric Theory*. N.Y.; L: Interscience; 1957.
5. Kauderer H. *Nichtlineare Mechanik*. Berlin: Springer; 1958.
6. Kharkevich AA. *Self-Excited Oscillations*. Moscow: Gostekhizdat; 1953.
7. Teodorchik KF. *Self-Excited Oscillatory Systems*. Moscow and Leningrad: Gostekhizdat; 1952.
8. Akulenko LD, Korovina LI, Nesterov SV. Self-excited oscillations of extremely non-linear systems. *Izv Ross Akad Nauk MTT* 2002;**3**:42–8.
9. Akulenko LD, Korovina LI, Kumakshev SA, Nesterov SV. Self-excited oscillations of Rayleigh and Van der Pol oscillators for moderately high feedback factors. *Prikl Mat Mekh* 2004;**68**(2):273–81.
10. Akulenko LD, Nesterov SV. An analytical investigation of the quasi-linear self-excited oscillations of a Helmholtz resonator. *Izv Ross Akad Nauk MZhG* 2004;**3**:6–15.
11. Mitropol'skii YuA, Lykova OB. *Integral Manifolds in Non-linear Mechanics*. Moscow: Nauka; 1973.

12. Akulenko LD. Periodic motions of autonomous systems in the neighbourhood of a stable integral manifold. *Izv Ross Akad Nauk MTT* 1995;**3**:26–35.
13. Bogolyubov NN, Mitropol'skii YuA. *Asymptotic Methods in the Theory of Non-linear Oscillations*. Moscow: Nauka; 1974.
14. Volosov VM, Morgunov BI. *The Averaging Method in the Theory of Non-linear Oscillatory Systems*. Moscow: Izd MGU; 1971.
15. Vasil'yeva AB, Butuzov VF. *Asymptotic Expansions of the Solutions of Singularly Perturbed Equations*. Moscow: Nauka; 1973.
16. Yerugin NP. *Implicit Functions*. Leningrad: Izd LGU; 1956.

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